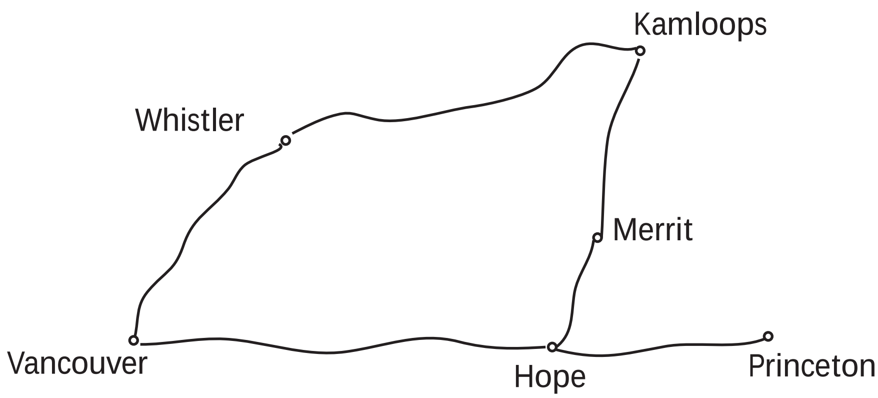
SECTIONS 1.5-1.6
NOTES ON GRAPH THEORY NOTATION AND ITS USE IN THE STUDY OF SPARSE SYMMETRIC MATRICES

A graph $G = (X, E)$ consists of a finite set of nodes or vertices X and edges E.

EXAMPLE : A road map of part of British Columbia

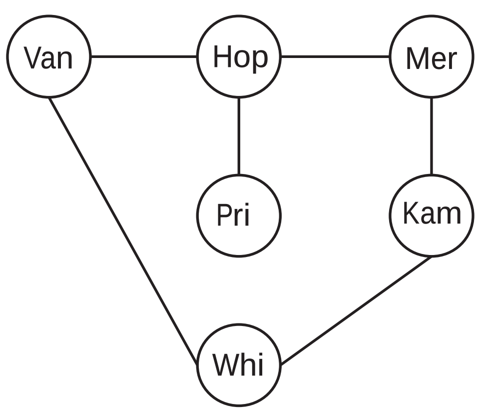


The information contained in this map can be represented by a graph with $N = 6$ vertices.

X = { Van, Whi, Kam, Hop, Pri, Mer }
E = { (Van, Whi), (Whi, Kam), (Hop, Van), (Kam, Mer), (Hop, Pri), (Mer, Hop) }

Note that E is a set of unordered pairs; that is, (Van, Whi) is the same edge as (Whi, Van).

Usually a graph is represented as follows (rather than by listing the sets X and E):



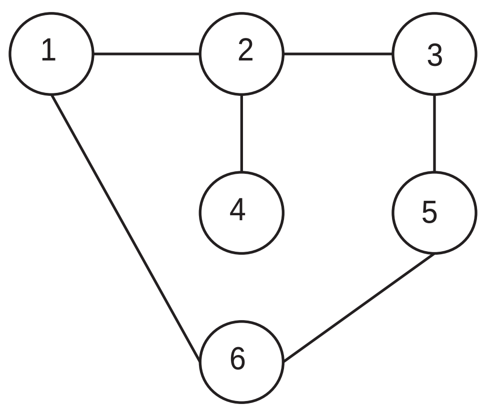
Note that although the above graph looks quite different from the map of B.C., all of the information contained in the sets X and E is retained.

The above is an example of an unordered (or unlabelled) graph.

An ordering (or labelling) α is a mapping of the integers $\{1, 2, \dots, N\}$ onto X. For example, if α is the mapping

$$
\begin{aligned}
1 &\rightarrow \text{Van} \\
2 &\rightarrow \text{Hop} \\
3 &\rightarrow \text{Mer} \\
4 &\rightarrow \text{Pri} \\
5 &\rightarrow \text{Kam} \\
6 &\rightarrow \text{Whi}
\end{aligned}
$$

then the above unordered graph becomes the ordered graph in Figure 3.1.1 :



The relationship between graphs on N nodes and $N \times N$ symmetric matrices:

> an $N \times N$ symmetric matrix A has an associated ordered graph with node set $X = \{1, 2, \dots, N\}$ and edge set E such that $(i, j) = (j, i) \in E$ if and only if $a_{ij} = a_{ji} \neq 0$ and $i \neq j$.

As we will be interested only in positive definite matrices, which always have all diagonal entries nonzero, we will put nonzeros in all positions on the main diagonal. Thus, as in Figure 3.1.1, the matrix A associated with the above graph has nonzeros in positions indicated by an * as follows:

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 & * \\ * & * & * & * & 0 & 0 \\ 0 & * & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & * & * \\ * & 0 & 0 & 0 & * & * \end{bmatrix}$$

(Note in the George/Liu notes, the diagonal entries are denoted by circled integers, rather than *.)

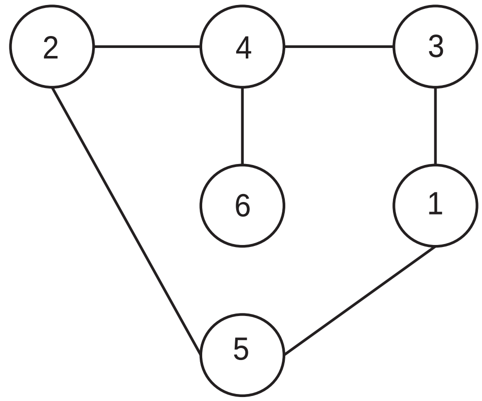
Let $P \neq I$ be an $N \times N$ permutation matrix. Then PAP^T is a symmetric reordering of the rows and columns of A. For example, in Figure 3.1.2,

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and (using the above matrix A)

$$PAP^T = \begin{bmatrix} * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & * & 0 \\ * & 0 & * & * & 0 & 0 \\ 0 & * & * & * & 0 & * \\ * & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \end{bmatrix}$$

and the associated graph for this matrix is



How do you determine the permutation matrix P such that for the original matrix A above and its associated graph, the graph associated with PAP^T is as above? The nonzeros in P can be determined as follows:

Mapping of the nodes	Nonzeros in P
$1 \to 2$	(2, 1)
$2 \to 4$	(4, 2)
$3 \to 3$	(3, 3)
$4 \to 6$	(6, 4)
$5 \to 1$	(1, 5)
$6 \to 5$	(5, 6)

The unlabelled (or unordered) graphs of A and PAP^T are the same – they represent the "structure" of A or the equivalence class of all matrices PAP^T where P is any $N \times N$ permutation matrix.

The ordered graphs are associated with PAP^T for different permutation matrices P. The problem of finding a "good" permutation matrix for A (with respect to some sparse matrix problem) is equivalent to finding a "good" ordering (or labeling) of the graph of A.

TERMINOLGY

Two nodes x and y are adjacent in a graph G if $(x, y) = (y, x) \in E$. In this case, the nodes x and y are said to be neighbors.

The adjacent set of a node y in G is

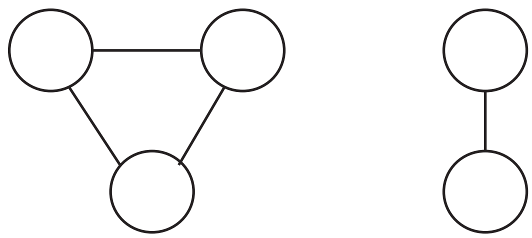
$$Adj(y) = \{x \in X : (x, y) \in E\}.$$

The degree of a node y is $|Adj(y)|$, the cardinality of the set $Adj(y)$.

A (simple) path from node x to node y of length $\ell \geq 1$ in G is an ordered set of $\ell + 1$ distinct nodes $(v_1, v_2, \ldots, v_{\ell+1})$ such that $v_{i+1} \in Adj(v_i)$, for $i = 1, 2, \ldots, \ell$ with $v_1 = x$ and $v_{\ell+1} = y$.

A graph G is connected if every pair of distinct nodes is joined by at least one path. Otherwise, G is disconnected and consists of two or more connected components.

A disconnected graph with two connected components:



Relationship with matrices:

the graph of a matrix A is disconnected if and only if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

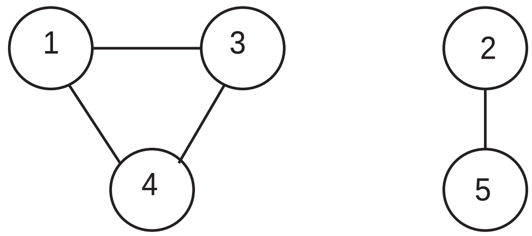
where A_{11} and A_{22} are square (nonempty) submatrices. Such a matrix PAP^T is called a block diagonal matrix.

EXAMPLE

Suppose that a matrix A has the following zero/nonzero structure:

$$A = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & * & 0 & 0 & * \\ * & 0 & * & * & 0 \\ * & 0 & * & * & 0 \\ 0 & * & 0 & 0 & * \end{bmatrix}.$$

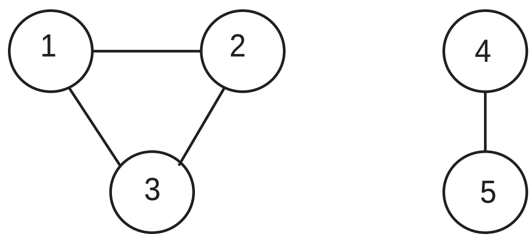
The graph of A is



With

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{we obtain} \quad PAP^T = \begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix},$$

which is a block diagonal matrix. The graph of PAP^T is



<u>Definition</u>

A symmetric matrix A is <u>reducible</u> if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonempty) submatrices. If such a permutation matrix P does not exist, then A is <u>irreducible</u>.

THEOREM

A symmetric matrix A is reducible if and only if its associated graph is disconnected. A symmetric matrix A is irreducible if and only if its associated graph is connected.